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Multiple Coupling of Topological Coherent Modes of Trapped Atoms

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Abstract

The possibility of generating multiple coherent modes in trapped Bose gases is advanced. This requires the usage of several driving fields whose frequencies are tuned close to the corresponding transition frequencies. A general criterion is derived explaining when the driving fields, even being in perfect resonance, cannot generate the topological coherent modes. This criterion is termed the theorem of shape conservation. Bose-Einstein condensates with generated coherent modes display a number of interesting effects, such as: interference fringes, interference current, mode locking, dynamic transition, critical phenomena, chaotic motion, harmonic generation, parametric conversion, atomic squeezing, and entanglement production. Approximate solutions, based on the averaging techniques, are found to be in good agreement with direct numerical calculations for the Gross-Pitaevskii equation.

1 Introduction

Dilute Bose gases at low temperatures are described by the Gross-Pitaevskii equation (see reviews [1–6]). The ground-state solution of the stationary Gross-Pitaevskii equation corresponds to the equilibrium Bose-Einstein condensate. In the presence of a trapping potential, this equation possesses a discrete spectrum and, respectively, a discrete set of wave functions describing the topological coherent modes, which represent nonground-state condensates [7]. These modes are called topological since the related wave functions have different spatial shapes, with different number of zeros. And the modes are termed coherent because the related wave functions correspond to coherent states [8]. Vortices are a particular case of such modes [9,10].

In practice, the topological coherent modes can be generated by subjecting the system of trapped atoms to the action of an alternating external field with a frequency tuned to the resonance with a transition frequency between the ground state and an excited collective level [7,9–12]. The properties of these modes have been studied in several publications [7–26]. Instead of modulating with a resonant field the trapping potential, one can periodically vary the atomic scattering length by invoking the Feshbach resonance [27–29]. Note that the coherent modes, being the solutions to the nonlinear Gross-Pitaevskii equation, in some cases can be considered as analytical continuations, under increasing nonlinearity, of the related linear modes; but there exist also the irreducible modes having no linear counterparts. The latter happens for sufficiently strong nonlinearity due to atomic interactions, when atoms are in double- or multiwell potentials [19,22] or inside an optical lattice [30–33]. Thermodynamics of the coherent modes has also been considered [34].

One should not confuse the topological coherent modes, which are the solutions to the nonlinear Gross-Pitaevskii equation, with the elementary excitations treated by the linear Bogolubov - De Gennes equations. In principle, one could consider resonant transitions between the modes of elementary excitations [35], which, however, is a radically different problem.

In previous publications, the case of a single generated coherent mode has been studied, which requires the action of one external resonant field. Here, we generalize the consideration to the multiple mode generation, which can be done by involving several alternating fields. It is particularly important that we have accomplished all investigations in two ways, by employing approximate analytical methods, based on the averaging techniques, and also by solving numerically the Gross-Pitaevskii equation, both ways being in good agreement with each other.

2 Multiple Mode Generation

Topological coherent modes are the solutions to the stationary Gross-Pitaevskii equation

$$\hat{H}[\varphi_n]\varphi_n(\mathbf{r}) = E_n\varphi_n(\mathbf{r}) \; , \tag{1}$$

with the nonlinear Hamiltonian

$$\hat{H}[\varphi] = -\,\frac{\hbar^2}{2m_0}\,\nabla^2 + U(\mathbf{r}) + NA_s|\varphi|^2 \tag{2}$$

containing a trapping potential $U(\mathbf{r})$ and the nonlinear term due to the binary contact interactions

$$A_s \equiv 4\pi\hbar^2 \, \frac{a_s}{m_0} \; ;$$

N being the number of particles in the trap; m_0, particle mass; a_s, scattering length. The functions $\varphi_n(\mathbf{r})$ are normalized to unity, $(\varphi_n, \varphi_n) = 1$. In general, φ_m and φ_n are not compulsorily orthogonal.

Temporal evolution is described by the time-dependent Gross-Pitaevskii equation

$$i\hbar \, \frac{\partial}{\partial t} \, \varphi(\mathbf{r}, t) = \left(\hat{H}[\varphi] + \hat{V} \right) \varphi(\mathbf{r}, t) \; , \tag{3}$$

with an external potential $\hat{V} = V(\mathbf{r}, t)$, which is assumed to have the multifrequency form

$$\hat{V} = \frac{1}{2} \, \sum_j \left[B_j(\mathbf{r}) e^{i\omega_j t} + B_j^*(\mathbf{r}) e^{-i\omega_j t} \right] \; , \tag{4}$$

in which $j = 1, 2, \ldots$. The frequencies of the alternating potential (4) are chosen so that each of them is tuned close to one of the transition frequencies

$$\omega_{mn} \equiv \frac{1}{\hbar} \, (E_m - E_n) \tag{5}$$

between two coherent modes, which implies the validity of the resonance conditions

$$\left| \frac{\Delta_{mn}}{\omega_{mn}} \right| \ll 1 \; , \qquad \Delta_{mn} \equiv \omega_j - \omega_{mn} \; . \tag{6}$$

Transitions between different modes can be produced by the action of the multiresonant potential (4) and also owing to interatomic interactions. Consequently, there are two types of transition amplitudes, defined as the matrix elements over the related coherent modes. The transition amplitudes

$$\alpha_{mn} \equiv N \, \frac{A_s}{\hbar} \left(|\varphi_n|^2, \; 2|\varphi_n|^2 - |\varphi_m|^2 \right) \tag{7}$$

correspond to interatomic interactions, while the transition amplitudes

$$\beta_{mn} \equiv \frac{1}{\hbar} \left(\varphi_m, \hat{B}_j \varphi_n \right) \; , \tag{8}$$

with $\hat{B}_j = B_j(\mathbf{r})$, are characterized by the related resonant parts of the potential (4). Good resonance can be supported provided the transition amplitudes (7) and (8) are much smaller than the related transition frequencies,

$$\left| \frac{\alpha_{mn}}{\omega_{mn}} \right| \ll 1 \; , \qquad \left| \frac{\beta_{mn}}{\omega_{mn}} \right| \ll 1 \; . \tag{9}$$

The first of these inequalities limits the number of atoms in the trap [12]. This limiting number of atoms is of the order of the critical number of atoms with attractive interactions

allowing for the stability of the trapped atomic gas [7,12,36]. Hence, the generation of coherent modes is admissible for atomic systems with repulsive as well as attractive interactions.

The solution to Eq. (3) can be presented as the expansion

$$\varphi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r}) \, \exp\left(-\, \frac{i}{\hbar}\, E_n t\right) \tag{10}$$

over the coherent modes. The coefficient functions $c_n(t)$ are assumed to be slowly varying in time as compared to the exponentials, so that

$$\frac{1}{\omega_{mn}} \left| \frac{dc_n}{dt} \right| \ll 1 \; . \tag{11}$$

Inequalities (8) and (11) are mutually self-consistent, since if $\alpha_{mn} \rightarrow 0$ and $\beta_{mn} \rightarrow 0$, then $c_n(t) \rightarrow const$ and $dc_n/dt \rightarrow 0$. Substituting expansion (10) into the evolution equation (3) and employing the averaging technique [7,12] results in the system of equations for the functions $c_n(t)$. The values $|c_n(t)|^2$ describe fractional mode populations. For instance, in the case of three modes coupled by two resonant fields, we find

$$i\, \frac{dc_1}{dt} = \left(\alpha_{12}|c_2|^2 + \alpha_{13}|c_3|^2\right) c_1 + f_1 \; ,$$

$$i\, \frac{dc_2}{dt} = \left(\alpha_{21}|c_1|^2 + \alpha_{23}|c_3|^2\right) c_2 + f_2 \; ,$$

$$i\, \frac{dc_3}{dt} = \left(\alpha_{31}|c_1|^2 + \alpha_{32}|c_2|^2\right) c_3 + f_3 \; , \tag{12}$$

where the terms f_j depend on the kind of the mode generation involved. Thus, for the cascade generation,

$$f_1 = \frac{1}{2}\, \beta_{12} c_2 e^{i\Delta_{21}t} \; , \qquad f_2 = \frac{1}{2}\, \beta_{12}^* c_1 e^{-i\Delta_{21}t} + \frac{1}{2}\, \beta_{23} c_3 e^{i\Delta_{32}t} \; ,$$

$$f_3 = \frac{1}{2}\, \beta_{23}^* c_2 e^{-i\Delta_{32}t} \; ; \tag{13}$$

for the V-type generation,

$$f_1 = \frac{1}{2}\, \beta_{12} c_2 e^{i\Delta_{21}t} + \frac{1}{2}\, \beta_{13} c_3 e^{i\Delta_{31}t} \; , \qquad f_2 = \frac{1}{2}\, \beta_{12}^* c_1 e^{-i\Delta_{21}t} \; ,$$

$$f_3 = \frac{1}{2}\, \beta_{13}^* c_1 e^{-i\Delta_{31}t} \; ; \tag{14}$$

and for the Λ-type generation

$$f_1 = \frac{1}{2}\, \beta_{13} c_3 e^{i\Delta_{31}t} \; , \qquad f_2 = \frac{1}{2}\, \beta_{23} c_3 e^{i\Delta_{32}t} \; ,$$

$$f_3 = \frac{1}{2}\, \beta_{13}^* c_1 e^{-i\Delta_{31}t} + \frac{1}{2}\, \beta_{23}^* c_2 e^{-i\Delta_{32}t} \; . \tag{15}$$

Equations (12) are to be complimented by initial conditions and by the normalization condition

$$|c_1|^2 + |c_2|^2 + |c_3|^2 = 1 \; .$$

Note that Eqs. (12) enjoy the global phase symmetry $c_j \rightarrow c_j \exp(i\chi)$, with $\chi \in [0, 2\pi]$. This could allow us to take one of the initial conditions for $c_j(0)$ as a real quantity. However, one cannot make all three functions $c_j(t)$ real. In general, they are complex-valued functions.

Let us emphasize that all generated modes coexist in the same trap and are not spatially separated. This distinguishes the considered situation, both physically as well as mathematically, from the cases of spatially separated condensates, as in multiwell potentials [37] or in clouds separated by means of the collective Rayleigh scattering [38–41]. The coherent modes, we consider, differ from each other by the shapes of their wave functions and by their energies, which are defined by the Gross-Pitaevskii equation (1).

3 Theorem of Shape Conservation

Applied alternating potentials not always can generate new topological coherent modes, even if the required resonance conditions are perfectly valid. It may occur that the initial atomic cloud, being subject to the action of resonant fields, oscillates in space without changing its shape, but no higher modes are excited. The criterion for such a behaviour is provided by the theorem below.

Let us consider Bose-condensed atoms in a trapping potential $U(\mathbf{r})$, with a driving field $V(\mathbf{r}, t)$. The corresponding coherent wave function $\varphi(\mathbf{r}, t)$ satisfies the temporal Gross-Pitaevskii equation (3). Since atoms are trapped, then the *trapping condition*

$$\varphi(\mathbf{r}, t) \rightarrow 0 \qquad (|\mathbf{r}| \rightarrow \infty) \tag{16}$$

is always valid for all $t \geq 0$. Assume that at the initial time the function $\varphi(\mathbf{r}, 0)$ represents a real mode

$$\varphi(\mathbf{r}, 0) = \varphi_0(\mathbf{r}) = \varphi_0^*(\mathbf{r}) \tag{17}$$

satisfying the stationary equation (1). We shall say that the density of atoms conserves its shape when the *shape-conservation condition*

$$|\varphi(\mathbf{r}, t)| = |\varphi(\mathbf{r} - \mathbf{a}, 0)| \tag{18}$$

holds true, where $\mathbf{a} = \mathbf{a}(t)$ defines the center-of-mass motion. The following criterion has been proven [42].

Theorem. The solution to the Gross-Pitaevskii equation (3), obeying conditions (16) and (17), satisfies the shape-conservation condition (18) if and only if the trapping potential is harmonic, while the driving field is linear in space, that is,

$$U(\mathbf{r}) = A_0 + \mathbf{A}_1 \cdot \mathbf{r} + \sum_{\alpha\beta} A_{\alpha\beta} r^\alpha r^\beta \; , \qquad V(\mathbf{r}) = B_0(t) + \mathbf{B}_1(t) \cdot \mathbf{r} \; , \tag{19}$$

where $B_0(t)$ and $\mathbf{B}_1(t)$ are arbitrary functions of time. Then the center-of-mass motion is given by the equation

$$m_0 \, \frac{d^2 a^\alpha}{dt^2} + \sum_\beta (A_{\alpha\beta} + A_{\beta\alpha}) a^\beta + B_1^\alpha(t) = 0 \; . \tag{20}$$

This theorem shows that, even when the functions $B_0(t)$ and $\mathbf{B}_1(t)$ in Eq. (19) correspond to alternating fields being in resonance with some transition frequencies, no generation of other modes is possible, but the density of atoms will conserve its shape, with the center-of-mass motion described by Eq. (20). Hence, in order to effectively generate coherent modes, one has to avoid condition (19).

4 Dynamic Resonant Effects

By applying k alternating fields, with the frequencies ω_j $(j = 1, 2, \ldots, k)$ one can generate k topological modes, starting with one given mode, say, having at the initial time all atoms in the ground state. Then we will get $k + 1$ coupled coherent modes. The mode amplitudes c_j, defining the fractional populations $|c_j|^2$, obey $k+1$ nonlinear differential equations. Because of their nonlinearity, these equations display a rich variety of interesting features. We have thoroughly examined the dynamics of c_j for the cases of two and three coupled coherent modes, finding all fixed points and accomplishing a complete stability analysis. For the corresponding physical cases, we have also solved the Gross-Pitaevskii equation numerically. Both approaches were found to be in good agreement with each other. Since the derivation of the equations for c_j is based on the averaging technique [43], the resulting equations are approximate, with the accuracy of their solutions characterized by the values of the involved parameters. The errors of such approximate solutions are of the order of $\max\{|\alpha_{mn}/\omega_{mn}|, |\beta_{mn}/\omega_{mn}|\}$. Within this accuracy, the results obtained by solving the evolution equations for the mode amplitudes c_j coincide with those following from the direct numerical solution of the Gross-Pitaevskii equation. Details of these calculations will be published in a separate paper. And here, being limited in space, we shall present a brief account of the most interesting physical effects we have found.

(1) *Interference Fringes*

The total density of atoms in the trap, $\rho(\mathbf{r}, t) \equiv |\varphi(\mathbf{r}, t)|^2$, is not equal to the sum of the partial densities $\rho_j(\mathbf{r}, t) \equiv |\varphi_j(\mathbf{r}, t)|^2$ where $\varphi_j(\mathbf{r}, t) = c_j(t)\varphi_j(\mathbf{r})$, but there exist the characteristic interference fringes described by the interference density

$$\rho_{int}(\mathbf{r}, t) \equiv \rho(\mathbf{r}, t) - \sum_j \rho_j(\mathbf{r}, t) \; . \tag{21}$$

The appearance of such interference patterns is typical of coexisting coherent states [12,20,25,26].

(2) *Interference Current*

Because of an essential nonuniformity of the atomic density for several coexisting coherent modes, there arises the interference current

$$\mathbf{j}_{int}(\mathbf{r}, t) \equiv \mathbf{j}(\mathbf{r}, t) - \sum_i \mathbf{j}_i(\mathbf{r}, t) \; , \tag{22}$$

which, similarly to the interference density (21), is the difference between the total current

$$\mathbf{j}(\mathbf{r}, t) \equiv \frac{\hbar}{m_0} \, \mathrm{Im} \, \varphi^*(\mathbf{r}, t)\vec{\nabla}\varphi(\mathbf{r}, t)$$

and the sum of the partial currents

$$\mathbf{j}_i(\mathbf{r}, t) \equiv \frac{\hbar}{m_0} \, \mathrm{Im} \, \varphi_i^*(\mathbf{r}, t) \vec{\nabla} \varphi_i(\mathbf{r}, t) \; .$$

The interference current is also typical of the case of different coexisting coherent modes [12,20,25,26]. Such a current is called sometimes the internal Josephson current or topological current. The possibility of the Josephson-type oscillations between two interpenetrating populations, not separated by any barrier, was suggested by Leggett [44].

(3) *Mode Locking*

When the amplitudes of the driving resonant fields are sufficiently small, such that $|\beta_{mn}/\alpha_{mn}| \ll 1$, the fractional mode populations exhibit nonlinear oscillations close to their initial values, never crossing the line 1/2, that is, depending on initial conditions, one has either

$$0 \leq |c_j|^2 \leq \frac{1}{2} \tag{23}$$

or

$$\frac{1}{2} \leq |c_j|^2 \leq 1 \; . \tag{24}$$

The mode locking effect exists for two [7,11,12] as well as for several [42] coupled modes. Mathematically, it is the same as self-trapping [45] of atoms in one of the wells of a stationary multiwell potential.

(4) *Dynamic Transition*

Increasing the amplitudes of the driving fields up to the values such that $|\beta_{mn}/\alpha_{mn}| \approx 0.5$, one passes from the mode-locked regime of motion to the mode-unlocked regime, when the mode populations oscillate in the whole region between zero and one,

$$0 \leq |c_j|^2 \leq 1 \; . \tag{25}$$

The change of this dynamic behaviour occurs when the starting point of a trajectory is crossed by a saddle separatrix [11,12,25,26]. A similar effect of the change from the self-trapped to untrapped regime exists for multiwell potentials [46].

(5) *Critical Phenomena*

On the manifold of the system parameters α_{mn}, β_{mn}, and Δ_{mn}, there exist surfaces whose crossing results in the dynamic transitions from the mode-locked to mode-unlocked regimes of motion. In the vicinity of these separating surfaces, the temporal behaviour of the mode populations experiences sharp changes [11,12,18,20,26], because of which the separating surface can be termed the critical surface. Moreover, the time-averaged system exhibits on a critical surface critical phenomena analogous to those happening in equilibrium statistical systems under second-order phase transitions [20].

(6) *Chaotic Motion*

When the amplitudes of the driving fields become so large that the ratio $|\beta_{mn}/\alpha_{mn}|$ is of order one, the temporal evolution of the fractional mode populations $|c_j|^2$ can go chaotic.

This takes place only when the number of the coupled coherent modes is equal or larger than three. Two coexisting modes display always a regular behaviour [11,12,26]. The onset of chaos for three or more modes is connected with the disappearance of stable fixed points [42]. This is similar to the arising chaos in a three-well potential [37].

(7) *Harmonic Generation*

In order to generate a coherent mode, it is not compulsory to invoke the standard resonance, as in Eq. (6), when the frequency ω of the driving field is close to the required transition frequency $\omega_{21} = (E_2 - E_1)/\hbar$. But, instead of the resonance condition $\omega = \omega_{21}$, one can employ one of the harmonic generation conditions

$$n\omega = \omega_{21} \qquad (n = 1, 2, \ldots) . \tag{26}$$

This type of harmonic generation is well known in quantum optics [47] and can also be realized for elementary excitations in Bose-Einstein condensates [48,49]. As we have shown [42], it exists as well for topological coherent modes.

(8) *Parametric Conversion*

When several driving fields are involved, with the frequencies ω_j $(j = 1, 2, \ldots)$, then there exist other possibilities for generating a coherent mode corresponding to the transition frequency ω_{21}. Thus, in the case of two driving fields, with the frequencies ω_1 and ω_2, the transition from a mode with energy E_1 to that of energy E_2 can be activated if $\omega_1 + \omega_2 = \omega_{21}$ or $\omega_1 - \omega_2 = \omega_{21}$. In the language of quantum optics [47], this is called parametric conversion, up or down, respectively. Generally, for several driving fields, the condition of parametric conversion can be written as

$$\sum_j (\pm \omega_j) = \omega_{21} . \tag{27}$$

An analogous effect is known in quantum optics [47] and for elementary excitations in trapped condensates [48,49]. Now, we proved [42] its existence for nonlinear coherent modes.

(9) *Atomic Squeezing*

Atomic squeezing can be defined by means of pseudospin operators, because of which it is often termed spin squeezing. For this purpose, one passes from the atomic operators a_i and $a_i^\dagger$, corresponding to the i-th mode and satisfying the commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$, to the pseudospin operators. In the case of the two-mode condensate, with $i = 1, 2$, one defines the collective spin operators

$$S_- = a_1^\dagger a_2 , \qquad S_+ = a_2^\dagger a_1 , \qquad S_z = \frac{1}{2}\left(a_2^\dagger a_2 - a_1^\dagger a_1 \right) ,$$

which satisfy the standard spin commutation relations

$$[S_+, S_-] = 2S_z , \qquad [S_z, S_\pm] = \pm S_\pm .$$

The amount of squeezing can be measured by the squeezing factor

$$Q_S \equiv \frac{2\Delta^2(S_z)}{| < S_\pm > |} , \tag{28}$$

where $\Delta^2(S_z)$ is the dispersion. Atomic operators are connected with the mode amplitudes by the equality $< a_i^\dagger a_j >= N c_i^* c_j$. Since $c_i = c_i(t)$ is a function of time, the squeezing factor (28) is an oscillating function [12].

For the 3-mode condensate, when $i = 1, 2, 3$, one introduces the collective angular momentum operators

$$J_- = \sqrt{2}\left(a_1^\dagger a_2 + a_2^\dagger a_3\right) , \qquad J_+ = \sqrt{2}\left(a_2^\dagger a_1 + a_3^\dagger a_2\right) , \qquad J_z = a_3^\dagger a_3 - a_1^\dagger a_1 ,$$

with the commutation relations

$$[J_+, J_-] = 2J_z , \qquad [J_z, J_\pm] = \pm J_\pm .$$

Then squeezing can be characterized by the squeezing factor

$$Q_J \equiv \frac{2\Delta^2(J_z)}{| < J_\pm > |} ,$$

similar to factor (28). The generation of atomic squeezing can be useful for spectroscopy.

(10) *Entanglement Production*

To quantify the amount of entanglement produced by a p-particle density matrix $\rho_p(t)$, we may employ the measure

$$\varepsilon(\rho_p) = \log \frac{||\rho_p||_{\mathcal{D}}}{||\rho_p^{\otimes}||_{\mathcal{D}}} , \tag{29}$$

where $||\rho_p||_{\mathcal{D}}$ implies the Hermitian norm over a set $\mathcal{D}$ of disentangled states and $\rho_p^{\otimes}$ means the disentangled counterpart of ρ_p (see details in [50]). For a multimode condensate, we have

$$\varepsilon(\rho_p) = (1-p) \log \sup_j |c_j|^2 . \tag{30}$$

The mode amplitudes $c_j = c_j(t)$ depend on time. Therefore measure (30) describes the evolutional entanglement. The temporal behaviour of the functions $c_j(t)$ can be regulated by switching on and off the resonant driving fields. Hence the measure (30) of the produced entanglement can also be regulated, which opens the possibility for information processing by means of the multimode resonant condensates.

5 Conclusion

We have shown that multiple topological coherent modes can be generated in a Bose-Einstein condensate, which requires the usage of several driving fields. It is important to emphasize that we have accomplished the consideration in two ways, by applying the averaging techniques and by direct numerical simulations for the Gross-Pitaevskii equation, both ways being in good agreement with each other. The multimode condensate presents a novel object possessing a variety of interesting properties. From one side it reminds us a multilevel resonant atom, so widely considered in quantum optics, but, at the same time, it is a multiatomic system, because of which it exhibits unusual nonlinear features. The multimode condensate displays the following remarkable effects: interference fringes, interference

current, mode locking, dynamic transition, critical phenomena, chaotic motion, harmonic generation, parametric conversion, atomic squeezing, and entanglement production. The rich variety of interesting properties, characteristic of the multimode condensates, suggests that the latter could find a number of applications.

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